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2009 J. Phys. A: Math. Theor. 42 345205

(http://iopscience.iop.org/1751-8121/42/34/345205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 345205 (18pp) doi:10.1088/1751-8113/42/34/345205

Integrable equations in 2 + 1 dimensions: deformations of dispersionless limits

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Received 15 May 2009, in final form 10 July 2009
Published 10 August 2009
Online at stacks.iop.org/JPhysA/42/345205

Abstract

We classify integrable third-order equations in 2 + 1 dimensions which generalize the examples of Kadomtsev–Petviashvili, Veselov–Novikov and Harry Dym equations. Our approach is based on the observation that dispersionless limits of integrable systems in 2 + 1 dimensions possess infinitely many multi-phase solutions coming from the so-called hydrodynamic reductions. In this paper, we adopt a novel perturbative approach to the classification problem. Based on the method of hydrodynamic reductions, we first classify integrable quasilinear systems which may (potentially) occur as dispersionless limits of soliton equations in 2 + 1 dimensions. To reconstruct dispersive deformations, we require that all hydrodynamic reductions of the dispersionless limit be inherited by the corresponding dispersive counterpart. This procedure leads to a complete list of integrable third-order equations, some of which are apparently new.

PACS numbers: 02.30.Ik, 02.30.Jr
Mathematics Subject Classification: 35L40, 35Q51, 35Q58, 37K10, 37K55

1. Introduction

The classification of integrable systems has been a topic of active research from the very beginning of soliton theory. In 1 + 1 dimensions, this resulted in extensive lists of integrable equations within particularly important subclasses [24], which were obtained by means of the symmetry approach. Although this technique generalizes to 2 + 1 dimensions, one encounters additional difficulties due to the appearance of non-local variables [25]. A way to bypass the problem of non-locality, known as the perturbative symmetry approach [26], provides an efficient way to classify soliton equations in 2 + 1 dimensions. In this framework, one starts with a linear equation having degenerate dispersion law [38], and reconstructs the allowed nonlinearity. However, few classification results have been obtained so far. In fact, most of the

(2 + 1)-dimensional examples known to date were derived by postulating a special structure of the corresponding Lax pair; see e.g. [21, 34].

In this paper, we adopt a novel approach to the problem of classification of scalar third-order soliton equations in 2 + 1 dimensions with the 'simplest' possible non-localities,

$$u_t = F(u, w, Du, Dw)$$

and

$$u_t = F(u, v, w, Du, Dv, Dw),$$

respectively. Here $u(x, y, t)$ is a scalar field, and the non-local variables $v(x, y, t)$ and $w(x, y, t)$ are defined via $w_x = u_y$ and $v_y = u_x$, equivalently, $w = D_x^{-1}D_y u$, $v = D_y^{-1}D_x u$. The symbols Du, Dv, Dw denote the collection of all partial derivatives of u, v, w with respect to x and y up to the third order. In fact, it is sufficient to allow only y-derivatives of w and x-derivatives of v. We will refer to the above equations as the 'non-symmetric' and 'symmetric' cases, respectively. We assume that in both cases the dependence of the right-hand side F on the derivatives of u and w (resp., u, v, w) is *polynomial*, where the coefficients are allowed to be arbitrary functions of u and w (resp., u, v, w). Explicitly, in the non-symmetric case we have

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\cdots) + \epsilon^2(\cdots), \qquad w_x = u_y, \tag{1}$$

where φ, ψ, η are the functions of u and w, while the terms at ϵ and ϵ^2 are assumed to be homogeneous differential polynomials of orders 2 and 3 in the derivatives of u and w, whose coefficients can be arbitrary functions of u and w. We use the following weighting scheme: u and w are assumed to have order 0, their derivatives u_x, u_y, w_x, w_y are of order 1, the expressions $u_{xx}, u_{xy}, u_{yy}, w_{yy}, u_x^2, u_x u_y, u_y^2, u_x w_y, u_y w_y, w_y^2$ are of order 2, etc. Thus, the term at ϵ is a linear combination of the ten second order expressions whose coefficients can be arbitrary functions of u and w. The most familiar example within class (1) is the Kadomtsev–Petviashvili (KP) equation,

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \qquad w_x = u_y.$$

Similarly, in the symmetric case we consider equations of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x + \epsilon(\cdots) + \epsilon^2(\cdots), \qquad w_x = u_y, v_y = u_x, \tag{2}$$

where $\varphi, \psi, \eta, \tau$ are the functions of u, v and w. A canonical example of the form (2) is the Veselov–Novikov (VN) equation,

$$u_t = (uv)_x + (uw)_y + \epsilon^2(u_{xxx} + u_{yyy}), \qquad w_x = u_y, v_y = u_x.$$

In section 2, we bring together other known examples of the form (1) and (2) which include the KP, VN, Harry Dym equations and their modifications.

Our approach to the classification problem is based on the following key observations.

- Dispersionless limits of integrable soliton equations in 2+1 dimensions possess infinitely many hydrodynamic reductions.

In particular, dispersionless limits of equations (1) and (2),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y \tag{3}$$

and

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x, \qquad w_x = u_y, v_y = u_x, \tag{4}$$

should possess infinitely many hydrodynamic reductions and, thus, must be integrable in the sense of [10]. It was observed in [10] that the method of hydrodynamic reductions provides an efficient classification criterion. Thus, as a first step, in section 3 we classify integrable first-order equations of the form (3) and (4) which may (potentially) occur as dispersionless limits of integrable equations of the form (1) and (2). We emphasize that the requirement of being a dispersionless limit of a *third-order* soliton equation imposes further severe constraints, so that very few particular cases obtained in section 3 do actually survive.

Given an integrable dispersionless equation, one needs to reconstruct dispersive deformations. In 1 + 1 dimensions, this problem has been a subject of extensive research in [7–9, 22]; see also [1]. In 2 + 1 dimensions, the reconstruction procedure is based on the following key observation [14]:

- Hydrodynamic reductions of dispersionless limits of integrable soliton equations can be deformed into reductions of the corresponding dispersive counterparts (strictly speaking, this is only true if the dispersionless limit is linearly non-degenerate; see section 4). Furthermore, the requirement of the inheritance of all hydrodynamic reductions allows one to efficiently reconstruct dispersive terms in 2 + 1 dimensions.

This suggests the following alternative *definition* of the integrability:

A (2 + 1)-dimensional system is said to be integrable if all hydrodynamic reductions of its dispersionless limit (which is assumed to be linearly non-degenerate) can be deformed into reductions of the corresponding dispersive counterpart.

Although this property is satisfied for all known integrable equations whose dispersionless limit is not totally linearly degenerate, it would be important to formulate more precise statements about the equivalence of our definition with more 'conventional' approaches to the integrability.

The procedure of the reconstruction of dispersive terms is thoroughly illustrated in section 4, where we examine case-by-case all integrable dispersionless limits from section 3. Our calculations result in a complete list of integrable (2 + 1)-dimensional equations, some of which are apparently new. It is important to emphasize that, although our approach is based on the requirement of the inheritance of hydrodynamic reductions, all examples from the final list do actually possess conventional Lax pairs. Altogether, we found three new equations. One of them is

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)], \qquad w_x = u_y, \tag{5}$$

where $B = \beta u D_x - D_y$, $\beta =$ const. It possesses the Lax pair

$$\psi_{xy} = \beta u \psi_{xx} + \frac{1}{3\epsilon^2}\psi, \qquad \psi_t = \epsilon^2\beta^3 u^3 \psi_{xxx} - \epsilon^2 \psi_{yyy} + 3\epsilon^2\beta^2 u u_y \psi_{xx} + \beta w \psi_x.$$

The second example is

$$u_t = \tfrac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + u w_y + \epsilon^2[B^3(u) - \beta u_x B^2(u)], \qquad w_x = u_y, \tag{6}$$

where again $B = \beta u D_x - D_y$, $\beta =$ const. The corresponding Lax pair is

$$\psi_{xy} = \beta u \psi_{xx} + \frac{1}{3\epsilon^2} u \psi,$$

$$\psi_t = \epsilon^2\beta^3 u^3 \psi_{xxx} - \epsilon^2\psi_{yyy} + 3\epsilon^2\beta^2 u u_y \psi_{xx} + \frac{\beta^2}{3} u^3 \psi_x + w\psi_y + \beta u u_y \psi.$$

We point out that similar Lax operators appeared in the context of the (2 + 1)-dimensional Camassa–Holm equation [39]. Our last example is a deformation of the Harry Dym (HD) equation,

$$u_t = \frac{\delta}{u^3}u_x - 2w u_y + u w_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}, \qquad w_x = u_y, \tag{7}$$

for $\delta = 0$ it reduces to the standard HD equation (example 9 of section 2.1). It has the Lax pair $L_t = [A, L]$, where

$$L = \frac{\epsilon^2}{u^2}D_x^2 + \frac{\epsilon}{\sqrt{3}}D_y + \frac{\delta}{4u^2},$$

$$A = \frac{4\epsilon^2}{u^3}D_x^3 + \left(-\frac{6\epsilon^2 u_x}{u^4} + \frac{2\sqrt{3}\epsilon w}{u^2}\right)D_x^2 + \frac{\delta}{u^3}D_x + \left(-\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2}\right).$$

All three examples belong to the non-symmetric case. In the symmetric case, we have no new equations apart from those listed in section 2.2. This leads to the following main result.

Theorem 1. *Equations (5)–(7) along with the known examples of KP, non-symmetric VN, HD equations and their modifications provide a complete list of integrable equations of the form (1) with $\eta \neq 0$ whose dispersionless limit is linearly non-degenerate:*

KP equation $\quad u_t = uu_x + w_y + \epsilon^2 u_{xxx},$

mKP equation $\quad u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx},$

Gardner equation $\quad u_t = \left(\beta w - \frac{\beta^2}{2}u^2 + \delta u\right)u_x + w_y + \epsilon^2 u_{xxx},$

VN equation $\quad u_t = (uw)_y + \epsilon^2 u_{yyy},$

mVN equation $\quad u_t = (uw)_y + \epsilon^2\left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y,$

HD equation $\quad u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$

deformed HD equation $\quad u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$

Equation (5) $\quad u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2\left[B^3(u) - \beta u_x B^2(u)\right],$

Equation (6) $\quad u_t = \frac{4}{3}\beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2\left[B^3(u) - \beta u_x B^2(u)\right].$

In the symmetric case, there exist only two examples of integrable equations of the form (2) with $\eta, \tau \neq 0$:

VN equation $\quad u_t = (uv)_x + (uw)_y + \epsilon^2 u_{xxx} + \epsilon^2 u_{yyy},$

mVN equation $\quad u_t = (uv)_x + (uw)_y + \epsilon^2\left(u_{xx} - \frac{3}{4}\frac{u_x^2}{u}\right)_x + \epsilon^2\left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y.$

The proof is summarized in section 4. Under the substitution $w = 0, u_y = 0$, equations (5) and (6) reduce to

$$u_t = \epsilon^2\beta^3\left(u^3 u_{xxx} + 3u^2 u_x u_{xx}\right) + \beta^2 u^2 u_x$$

and

$$u_t = \epsilon^2\beta^3\left(u^3 u_{xxx} + 3u^2 u_x u_{xx}\right) + \tfrac{4}{3}\beta^2 u^3 u_x,$$

respectively. In this form, they have appeared in [35]; see also [24] and references therein. It was pointed out (see, e.g., [18, 24, 29]) that there exist differential substitutions bringing these equations to a constant separant form. It would be interesting to find out whether equations (5)–(7) are related to any of the known soliton hierarchies: the main problem here is that the above differential substitutions do not extend to 2 + 1 dimensions in any obvious way.

Remark 1. The examples of VN and mVN equations show that different (2 + 1)-dimensional equations may have one and the same dispersionless limit.

Remark 2. Our approach to the classification problem does not apply to non-symmetric equations with $\eta = 0$ (or symmetric equations with $\eta = \tau = 0$). As we explain in section 3, these conditions are equivalent to the reducibility of the dispersion relations of the corresponding systems (3), (4). A familiar example within this class is the so-called 'breaking soliton' equation,

$$u_t = 2wu_x + 4uu_y - \epsilon^2 u_{xxy}, \qquad w_x = u_y,$$

see, e.g., [3]. Here $\varphi = 2w$, $\psi = 4u$, $\eta = 0$. Equations of this type are not amenable to the method of hydrodynamic reductions, and require an alternative approach.

2. Known examples

2.1. Non-symmetric case

Here, we bring together known examples of soliton equations whose dispersionless limit is of the form (3). The relation $w_x = u_y$ will be automatically assumed whenever w appears explicitly in the equation. Examples 1–6 list third-order equations. Examples 7–10 correspond to equations of order 5 or differential-difference equations.

Example 1. The KP equation,

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \tag{8}$$

arises in mathematical physics as a two-dimensional generalization of the KdV equation. Its dispersionless limit (the dKP equation),

$$u_t = uu_x + w_y, \tag{9}$$

also known as the Khokhlov–Zabolotskaya equation [36], is of interest in its own, playing important role in nonlinear acoustics, gas dynamics and differential geometry.

Example 2. The modified KP (mKP) equation,

$$u_t = (w - u^2/2)u_x + w_y + \epsilon^2 u_{xxx}, \tag{10}$$

has the dispersionless limit

$$u_t = (w - u^2/2)u_x + w_y. \tag{11}$$

Example 3. The (2 + 1)-dimensional version of the Gardner equation is of the form [21],

$$u_t = \left(\beta w - \frac{\beta^2}{2}u^2 + \delta u\right) u_x + w_y + \epsilon^2 u_{xxx}, \tag{12}$$

which reduces to the KP or mKP equations upon setting $\beta = 0$ or $\delta = 0$, respectively. Its dispersionless limit has the form

$$u_t = \left(\beta w - \frac{\beta^2}{2}u^2 + \delta u\right) u_x + w_y. \tag{13}$$

Example 4. The non-symmetric version of the VN equation [4, 28, 33],

$$u_t = (uw)_y + \epsilon^2 u_{yyy}, \tag{14}$$

has the dispersionless limit

$$u_t = (uw)_y. \tag{15}$$

Example 5. The non-symmetric version of the mVN equation [2],

$$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u} \right)_y, \tag{16}$$

has the same dispersionless limit as in the previous example,

$$u_t = (uw)_y. \tag{17}$$

Example 6. The HD equation [21],

$$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx}, \tag{18}$$

(set $\tilde{u} = 1/u$ to obtain the equation from [21]), has the dispersionless limit

$$u_t = -2wu_y + uw_y. \tag{19}$$

Example 7. The fifth-order version of the HD equation is

$$u_t = -3wu_y + uw_y - \frac{\epsilon^2}{u^4}\left(u^2 u_{xxy} - 3u(u_x u_y)_x + 6u_x^2 u_y\right) + \frac{\epsilon^4}{u^2}\left(\frac{1}{u^2}\right)_{xxxxx}; \tag{20}$$

see [21]. Its dispersionless limit has the form

$$u_t = -3wu_y + uw_y. \tag{21}$$

Example 8. The Toda lattice is a system of two differential-difference equations

$$\begin{aligned} \epsilon u_t &= u(w(y) - w(y - \epsilon)), \\ \epsilon w_x &= u(y + \epsilon) - u(y) \end{aligned} \tag{22}$$

or

$$\begin{aligned} u_t/u &= w_y - \frac{\epsilon}{2}w_{yy} + \frac{\epsilon^2}{6}w_{yyy} + \cdots + (-1)^{n+1}\frac{\epsilon^n}{n!}w_{ny} + \cdots, \\ w_x &= u_y + \frac{\epsilon}{2}u_{yy} + \frac{\epsilon^2}{6}u_{yyy} + \cdots + \frac{\epsilon^n}{n!}u_{ny} + \cdots. \end{aligned} \tag{23}$$

Its dispersionless limit is

$$u_t = uw_y. \tag{24}$$

Example 9. The non-local Toda lattice equation is

$$\epsilon \sigma_{xt} = \mathrm{e}^{\frac{\sigma(x+\epsilon, y+\epsilon) - \sigma}{\epsilon}} - \mathrm{e}^{\frac{\sigma - \sigma(x-\epsilon, y-\epsilon)}{\epsilon}}; \tag{25}$$

see [32]. Its dispersionless limit is

$$\sigma_{xt} = \mathrm{e}^{\sigma_x + \sigma_y}(\sigma_{xx} + 2\sigma_{xy} + \sigma_{yy}), \tag{26}$$

or, setting $\sigma_x = u, \sigma_y = w$,

$$u_t = \mathrm{e}^{u+w}(u_x + 2u_y + w_y).$$

Example 10. The BKP and CKP equations are of the forms

$$u_t - 5(u^2 + w)u_x - 5uw_x + 5w_y + \epsilon^2(uu_{xxx} + w_{xxx} + u_{xxx}) - \frac{\epsilon^4}{25}u_{xxxxx} = 0 \qquad (27)$$

and

$$u_t - 5(u^2 + w)u_x - 5uw_x + 5w_y + \epsilon^2\left(uu_{xxx} + w_{xxx} + \frac{5}{2}u_{xxx}\right) - \frac{\epsilon^4}{25}u_{xxxxx} = 0, \qquad (28)$$

respectively [21]. Their dispersionless limits coincide:

$$u_t = 5(u^2 + w)u_x + 5uu_y - 5w_y. \qquad (29)$$

2.2. Symmetric case

Here, we list known examples of the form (2). The relations $v_y = u_x$ and $w_x = u_y$ will be automatically assumed whenever v and w appear explicitly in the equation. It is quite remarkable that the 'symmetric' list is very restrictive and contains only two examples.

Example 1. The VN equation,

$$u_t = (uv)_x + (uw)_y + \epsilon^2 u_{xxx} + \epsilon^2 u_{yyy}, \qquad (30)$$

was introduced in [28, 33]. It has the dispersionless limit

$$u_t = (uv)_x + (uw)_y. \qquad (31)$$

Example 2. The mVN equation,

$$u_t = (uv)_x + (uw)_y + \epsilon^2\left(u_{xx} - \frac{3}{4}\frac{u_x^2}{u}\right)_x + \epsilon^2\left(u_{yy} - \frac{3}{4}\frac{u_y^2}{u}\right)_y, \qquad (32)$$

was first introduced in [2] (in a somewhat different form). It has the same dispersionless limit as in the previous example,

$$u_t = (uv)_x + (uw)_y. \qquad (33)$$

3. Classification of integrable dispersionless limits

In this section, we classify integrable dispersionless equations of the form (3) and (4) which may potentially occur as dispersionless limits of integrable soliton equations of the form (1) and (2), respectively. The integrability conditions are derived on the basis of the method of hydrodynamic reductions. For the convenience of the reader, we briefly recall the main steps of this construction. As proposed in [10], the method of hydrodynamic reductions applies to quasilinear equations of the following general form:

$$A(\mathbf{u})\mathbf{u}_t + B(\mathbf{u})\mathbf{u}_x + C(\mathbf{u})\mathbf{u}_y = 0; \qquad (34)$$

here $\mathbf{u} = (u^1, \ldots, u^m)^t$ is an m-component column vector of the dependent variables and A, B, C are $m \times m$ matrices. The method of hydrodynamic reductions consists of seeking multi-phase solutions in the form

$$\mathbf{u} = \mathbf{u}(R^1, \ldots, R^N), \qquad (35)$$

where the 'phases' $R^i(x, y, t)$ are required to satisfy a pair of consistent equations of hydrodynamic type,

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x.$$

We recall that the consistency conditions, $R^i_{yt} = R^i_{ty}$, imply the following restrictions for the characteristic speeds μ^i and λ^i:

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i},$$

$i \neq j, \partial_i = \partial/\partial_{R^i}$; see [31]. The substitution of the ansatz (35) into (34) leads to a complicated over-determined system of PDEs for functions $\mathbf{u}(R)$, $\mu^i(R)$ and $\lambda^i(R)$ whose coefficients depend on the matrix elements of A, B, C, and their derivatives. In particular, the characteristic speeds $\mu^i(R)$ and $\lambda^i(R)$ satisfy an algebraic relation $\det(\lambda A + B + \mu C) = 0$ which is nothing but the dispersion relation of the system (34). We will assume that the dispersion relation defines an irreducible algebraic curve of degree m.

Definition. [10]. *System (34) is said to be integrable if, for any number of phases N, it possesses infinitely many N-phase solutions parametrized by* $2N$ *arbitrary functions of one variable.*

The requirement of the existence of such solutions imposes strong constraints on the matrices A, B, C, which can be effectively computed. Although these constraints are quite formidable in general, there exists a simple necessary condition for the integrability which can be expressed in an invariant differential geometric form as follows. Let us first introduce the $m \times m$ matrix

$$V = (\alpha A + \beta B + \gamma C)^{-1}(\tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} C),$$

where α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are arbitrary constants. Given a (1, 1)-tensor $V = \left[v^i_j\right]$, let us introduce the following objects: Nijenhuis tensor

$$\mathcal{N}^i_{jk} = v^p_j \partial_{u^p} v^i_k - v^p_k \partial_{u^p} v^i_j - v^i_p \left(\partial_{u^j} v^p_k - \partial_{u^k} v^p_j\right),$$

Haantjes tensor

$$\mathcal{H}^i_{jk} = \mathcal{N}^i_{pr} v^p_j v^r_k - \mathcal{N}^p_{jr} v^i_p v^r_k - \mathcal{N}^p_{rk} v^i_p v^r_j + \mathcal{N}^p_{jk} v^i_r v^r_p.$$

One has the following result.

Theorem 2. [12]. *The vanishing of the Haantjes tensor is a necessary condition for the integrability of the system (34).*

Since the Haantjes tensor can be obtained using computer algebra, one gets an efficient integrability test (note that all components of the Haantjes tensor have to vanish for *any* values of the constants α, β, γ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$). These necessary conditions are very strong indeed, and in many cases turn out to be sufficient. We point out that, for $m = 2$, the Haantjes tensor vanishes identically and does not produce any non-trivial integrability conditions. In this case, one proceeds as follows: let us multiply (34) by A^{-1}, and diagonalize B (this is always possible in the 2-component case). Thus, without any loss of generality one can assume

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

In this particular normalization, the integrability conditions for 2×2 systems were obtained in [11]. These conditions constitute a system of second-order constraints for coefficients a, b, p, q, r, s which can easily be tested. Let us now apply this approach to the classification of integrable systems of the form (3) and (4).

3.1. Non-symmetric dispersionless limits

Given an equation of the form (3),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y,$$

let us first rewrite it in the matrix form (34) as follows:

$$\begin{pmatrix} -1/\varphi & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_x + \begin{pmatrix} \psi/\varphi & \eta/\varphi \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_y = 0.$$

This system is now in the form as studied in [11]. The integrability conditions reduce to a system of second-order partial differential equations for coefficients φ, ψ and η, which can be derived from the general integrability conditions for 2×2 systems of hydrodynamic type in $2 + 1$ dimensions as obtained in [11]:

$$\begin{aligned}
\varphi_{uu} &= -\frac{\varphi_w^2 + \psi_u \varphi_w - 2\psi_w \varphi_u}{\eta}, \\
\varphi_{uw} &= \frac{\eta_w \varphi_u}{\eta}, \\
\varphi_{ww} &= \frac{\eta_w \varphi_w}{\eta}, \\
\psi_{uu} &= \frac{-\varphi_w \psi_w + \psi_u \psi_w - 2\varphi_w \eta_u + 2\eta_w \varphi_u}{\eta}, \\
\psi_{uw} &= \frac{\eta_w \psi_u}{\eta}, \\
\psi_{ww} &= \frac{\eta_w \psi_w}{\eta}, \\
\eta_{uu} &= -\frac{\eta_w (\varphi_w - \psi_u)}{\eta}, \\
\eta_{uw} &= \frac{\eta_w \eta_u}{\eta}, \\
\eta_{ww} &= \frac{\eta_w^2}{\eta};
\end{aligned} \tag{36}$$

we assume $\eta \neq 0$: this is equivalent to the requirement that the dispersion relation of the system (3) define an irreducible conic (indeed, the condition $\det(\lambda A + B + \mu C) = 0$ is equivalent to $\lambda = \varphi + \psi\mu + \eta\mu^2$). We have verified that the system (36) is in involution, and all dispersionless limits appearing in section 2.1 indeed satisfy these integrability conditions. Equations (36) are straightforward to solve. First of all, the equations for η imply that, up to translations and rescalings, $\eta = 1, \eta = u$ or $\eta = \mathrm{e}^w h(u)$. We will consider all three possibilities case-by-case below. Note that φ and ψ are defined up to additive constants which can always be set equal to zero via the Galilean transformations of the initial equation (3). Moreover, the system (36) is form-invariant under transformations of the form

$$\tilde\varphi = \varphi - s\psi + s^2\eta, \qquad \tilde\psi = \psi - 2s\eta, \qquad \tilde\eta = \eta, \qquad \tilde u = u, \qquad \tilde w = w + su, \tag{37}$$

$s = \text{const}$, which correspond to the following transformations preserving the structure of equations (3):

$$\tilde x = x - sy, \qquad \tilde y = y, \qquad \tilde u = u, \qquad \tilde w = w + su.$$

All our classification results are formulated modulo this equivalence.

Case 1: $\eta = 1$. Then the remaining equations imply $\psi = \alpha w + f(u)$, $\varphi = \beta w + g(u)$, where f and g satisfy the linear ODEs

$$f'' = \alpha(f' - \beta), \qquad g'' = 2\alpha g' - \beta f' - \beta^2.$$

The subcase $\alpha = 0$ leads to polynomial solutions of the form

$$\psi = \gamma u, \qquad \varphi = \beta w - \tfrac{1}{2}\beta(\beta + \gamma)u^2 + \delta u. \tag{38}$$

Up to equivalence transformations, the case $\alpha \neq 0$ leads to exponential solutions,

$$\psi = \alpha w + \gamma \mathrm{e}^{\alpha u}, \qquad \varphi = \delta \mathrm{e}^{2\alpha u}; \tag{39}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants.

Case 2: $\eta = u$. Then the remaining equations imply $\psi = \alpha w + f(u)$, $\varphi = \beta w + g(u)$, where f and g satisfy the linear ODEs

$$uf'' = \alpha(f' - \beta) - 2\beta, \qquad ug'' = 2\alpha g' - \beta f' - \beta^2.$$

The case $\alpha \notin \{0, -1, -1/2\}$ leads to power-like solutions of the form

$$\psi = \alpha w + \gamma u^{\alpha+1}, \qquad \varphi = \delta u^{2\alpha+1}. \tag{40}$$

The subcase $\alpha = 0$ leads to logarithmic solutions,

$$\psi = -2\beta u \ln u - \beta u, \qquad \varphi = \beta w + \beta^2 u \ln^2 u + \delta u. \tag{41}$$

The subcase $\alpha = -1$ gives

$$\psi = -w + \gamma \ln u, \qquad \varphi = \delta/u. \tag{42}$$

Finally, the subcase $\alpha = -1/2$ gives

$$\psi = -\tfrac{1}{2}w + \gamma\sqrt{u}, \qquad \varphi = \delta \ln u. \tag{43}$$

Case 3: $\eta = \mathrm{e}^w h(u)$. Then the remaining equations imply $\psi = \mathrm{e}^w f(u)$, $\varphi = \mathrm{e}^w g(u)$, where f, g and h satisfy the nonlinear system of ODEs

$$h'' = f' - g, \qquad g''h = 2fg' - gf' - g^2, \qquad f''h = 2hg' - 2gh' + ff' - fg.$$

Setting $g = p'$, $f = h' + p$, we can rewrite this system as a pair of third-order ODEs

$$hp''' = 2h'p'' - p'h'' + 2pp'' - 2p'^2, \qquad hh''' = h'h'' - 2h'p' + hp'' + ph'',$$

which, up to a change of sign $p \to -p$, identically coincides with a system arising in the classification of integrable conservative hydrodynamic chains (subcase I_1 of section 3.1 in [13]). Setting $p = h'$, the second equation will be satisfied identically, while the first one implies a fourth-order ODE for h, $h''''h + 3(h'')^2 - 4h'h''' = 0$, whose general solution is an elliptic sigma-function: $h = \sigma(u)$, here $(\ln\sigma)'' = -\wp$, $(\wp')^2 = 4\wp^3 - c$ (note that $g_2 = 0$, $g_3 = c$). Thus, as a particular case we have

$$h = \sigma(u), \qquad f = 2\sigma'(u), \qquad g = \sigma''(u).$$

Another subclass of solutions can be obtained by setting $p = ch$ which implies

$$h'''h - h''h' = 2c(h''h - h'^2)$$

with the general solution

$$h = \alpha \mathrm{e}^{(c+\gamma)u} + \beta \mathrm{e}^{(c-\gamma)u};$$

here α, β, γ are arbitrary constants. Although the structure of the general solution is quite complicated, one can show that Case 3 cannot arise as a dispersionless limit of an integrable third-order soliton equation.

3.2. Symmetric dispersionless limits

In this section, we consider first-order equations of the form (4),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x, \qquad w_x = u_y, \qquad v_y = u_x,$$

where the coefficients $\varphi, \psi, \eta, \tau$ are the functions of u, v, w. We assume that the dispersion relation of this system defines an irreducible cubic, which is equivalent to the requirement $\eta \neq 0$ and $\tau \neq 0$ (indeed, the dispersion relation has the form $\lambda\mu = \tau + \varphi\mu + \psi\mu^2 + \eta\mu^3$). In this case, the integrability conditions reduce to a system of first-order partial differential equations for coefficients φ, ψ, η and τ which can be obtained from the requirement of the vanishing of the Haantjes tensor [12] as outlined in section 3. The details are as follows: first we rewrite equation (4) in the matrix form,

$$A\mathbf{u}_t + B\mathbf{u}_x + C\mathbf{u}_y = 0,$$

where $\mathbf{u}$ is a 3-component column vector $\mathbf{u} = (u, v, w)^t$, and A, B, C are 3×3 matrices,

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \varphi & \tau & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \psi & 0 & \eta \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The necessary conditions for integrability can be obtained from the requirement of the vanishing of the Haantjes tensor of the following family of matrices:

$$(\alpha A + \beta B + \gamma C)^{-1}(\tilde{\alpha} A + \tilde{\beta} B + \tilde{\gamma} C).$$

In fact, it is sufficient to require the vanishing of the Haantjes tensor for a 2-parameter family $(\alpha A + B)^{-1}(\tilde{\alpha} A + C)$. This condition turns out to be very restrictive, and leads to the following constraints for coefficients φ, ψ, η and τ:

$$\begin{aligned} \tau_u &= \varphi_v, & \eta_u &= \psi_w, \\ \tau_v &= \frac{\tau}{\eta}\psi_u, & \eta_v &= 0, \\ \tau_w &= 0, & \eta_w &= \frac{\eta}{\tau}\varphi_u, \\ \psi_v &= \varphi_w = 0, & \tau\psi_w &= \eta\varphi_v. \end{aligned}$$

The integration of this system is straightforward. First of all, one can set $\psi = f_u, \eta = f_w$ and $\varphi = g_u, \tau = g_v$ where $f = f(u, w)$ and $g = g(u, v)$. The separation of variables leads to the relations

$$\begin{aligned} f_w &= a(w)k(u), & g_v &= b(v)k(u), \\ f_{uu} &= \beta a(w)k(u), & g_{uu} &= \alpha b(v)k(u), \end{aligned}$$

where the functions $a(w), b(v)$ and $k(u)$ satisfy the ODEs $a' = \alpha a$, $b' = \beta b$ and $k'' = \alpha\beta k$; here α and β are arbitrary constants. Up to elementary translations, rescalings and Galilean transformations, this leads to the following subcases:

Case 1. $\alpha = \beta = 0$. This leads to equations of the form

$$u_t = \nu(uv)_x + \mu(uw)_y,$$

where μ, ν are arbitrary constants. These correspond to the Veselov–Novikov cases from section 2.2.

Case 2. $\alpha \neq 0, \beta = 0$. This leads to equations of the form

$$u_t = \nu(uv + \alpha u^3/6)_x + \mu(\mathrm{e}^{\alpha w}u)_y$$

and

$$u_t = \nu(v + \alpha u^2/2)_x + \mu(e^{\alpha w})_y,$$

where μ, ν, α are arbitrary constants.

Case 3. $\alpha \neq 0, \beta \neq 0$. This leads to equations of the form

$$u_t = \nu(e^{\beta v} k(u))_x + \mu(e^{\alpha w} k(u))_y,$$

where ν, μ, α, β are arbitrary constants, and $k'' = \alpha\beta k$.

4. Classification of integrable third-order dispersive equations

Given an integrable dispersionless limit, one has to reconstruct dispersive terms. This can be done by requiring that all hydrodynamic reductions of the dispersionless system be inherited by its dispersive counterpart. We will illustrate this procedure using the KP equation,

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx}, \qquad w_x = u_y.$$

Its dispersionless limit, the dKP equation,

$$u_t = uu_x + w_y, \qquad w_x = u_y,$$

possesses one-phase solutions of the form $u = R, w = w(R)$ where the phase $R(x, y, t)$ satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \qquad R_t = (\mu^2 + R)R_x; \tag{44}$$

here $\mu(R)$ is an arbitrary function, and $w' = \mu$. Equivalently, one can say that equations (44) constitute a 1-component hydrodynamic reduction of the dKP equation. Although the dKP equation is known to possess infinitely many N-component reductions for arbitrary N [15–17, 19], 1-component reductions will be sufficient for our purposes. The main observation of [14] is that *all* 1-component reductions (44) can be deformed into reductions of the full KP equation by adding appropriate dispersive terms which are *polynomial* in the x-derivatives of R. Explicitly, one has the following formulae for the deformed 1-phase solutions:

$$u = R, \qquad w = w(R) + \epsilon^2\left(\mu' R_{xx} + \tfrac{1}{2}(\mu'' - (\mu')^3)R_x^2\right) + O(\epsilon^4), \tag{45}$$

note that one can always assume that u remains undeformed modulo the Miura group [7]. The deformed equations (44) take the form

$$\begin{aligned} R_y &= \mu R_x + \epsilon^2\left(\mu' R_{xx} + \tfrac{1}{2}(\mu'' - (\mu')^3)R_x^2\right)_x + O(\epsilon^4), \\ R_t &= (\mu^2 + R)R_x + \epsilon^2\left((2\mu\mu' + 1)R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2)R_x^2\right)_x + O(\epsilon^4). \end{aligned} \tag{46}$$

In other words, the KP equation can be 'decoupled' into a pair of $(1 + 1)$-dimensional equations (46) in infinitely many ways, indeed, $\mu(R)$ is an arbitrary function. The series in (45) and (46) contain only even powers of ϵ, and do not terminate in general.

Conversely, the requirement of the inheritance of all 1-component reductions allows one to reconstruct dispersive terms: given the dKP equation, let us look for a third-order dispersive extension in the form

$$u_t = uu_x + w_y + \epsilon(\cdots) + \epsilon^2(\cdots), \qquad w_x = u_y, \tag{47}$$

where the terms at ϵ and ϵ^2 are homogeneous differential polynomials in the x- and y-derivatives of u and w of orders 2 and 3, respectively, whose coefficients are allowed to be arbitrary functions of u and w. We require that all 1-component reductions (44) can be deformed accordingly, so that we have the following analogues of equations (45) and (46),

$$u = R, \qquad w = w(R) + \epsilon(\cdots) + \epsilon^2(\cdots) + O(\epsilon^3) \tag{48}$$

and

$$R_y = \mu R_x + \epsilon(\cdots) + \epsilon^2(\cdots) + O(\epsilon^3),$$
$$R_t = (\mu^2 + R)R_x + \epsilon(\cdots) + \epsilon^2(\cdots) + O(\epsilon^3), \tag{49}$$

respectively. In equations (48) and (49), dots denote terms which are polynomial in the derivatives of R. Substituting equations (48) into (47) and using (49) along with the consistency conditions $R_{ty} = R_{yt}$, one arrives at a complicated set of relations allowing one to uniquely reconstruct dispersive terms in (47): not surprisingly, we obtain that all terms at ϵ vanish, while the terms at ϵ^2 result in the familiar KP equation. Moreover, one only needs to perform calculations up to the order ϵ^4 to arrive at this result! It is important to emphasize that the above procedure is required to work for *arbitrary* μ: whenever one obtains a differential polynomial in μ which has to vanish due to the consistency conditions, all its coefficients have to be set equal to zero independently. Another observation is that the reconstruction procedure does not necessarily lead to a unique dispersive extension as in the dKP case: one and the same dispersionless system may possess essentially non-equivalent dispersive extensions. In most of the cases, one can get the necessary classification results working with 1-component reductions only. There is, however, one particular situation where 1-component reductions are not sufficient. This is explained in the remark below.

Remark 1. Let us consider the dKP equation,

$$u_t = uu_x + w_y, \qquad w_x = u_y;$$

its 1-component reductions (44) can be shown to satisfy a pair of additional first-order constraints,

$$u_y^2 - u_x w_y = 0, \qquad (w_t - uu_y)u_x - u_y w_y = 0.$$

Conversely, any solution satisfying these constraints comes from 1-component reductions. Similarly, one can show that 2-component reductions of dKP are characterized by a pair of second-order differential constraints, etc. Let us introduce an extension of dKP in the form

$$u_t = uu_x + w_y + \epsilon\left(u_y^2 - u_x w_y\right), \qquad w_x = u_y;$$

by construction, it inherits all *undeformed* 1-component reductions: the ϵ-term vanishes on 1-component reductions identically. This extension is, however, not integrable: one can show that it is not consistent with the requirement of the inheritance of N-component reductions for $N \geqslant 2$. Thus, in what follows we eliminate deformations which inherit undeformed 1-component reductions.

In general, we proceed as follows. For definiteness, we will outline the algorithm for integrable dispersionless equations of the form (3),

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \qquad w_x = u_y.$$

Its 1-component reductions are of the form $u = R, w = w(R)$ where $R(x, y, t)$ satisfies a pair of Hopf-type equations

$$R_y = \mu R_x, \qquad R_t = (\varphi + \psi\mu + \eta\mu^2)R_x;$$

here $\mu(R)$ is an arbitrary function, and $w' = \mu$. We seek a third-order dispersive deformation of equation (3) in the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\cdots) + \epsilon^2(\cdots), \qquad w_x = u_y,$$

and postulate that 1-phase solutions can be deformed accordingly,

$$u = R, \qquad w = w(R) + \epsilon(\cdots) + \epsilon^2(\cdots) + O(\epsilon^3),$$

where

$$R_y = \mu R_x + \epsilon(\cdots) + \epsilon^2(\cdots) + O(\epsilon^3),$$
$$R_t = (\varphi + \psi\mu + \eta\mu^2)R_x + \epsilon(\cdots) + \epsilon^2(\cdots) + O(\epsilon^3).$$

Proceeding as outlined above we reconstruct possible dispersive terms. In fact, one can start with arbitrary φ, ψ, η: our procedure will eventually recover the constraints obtained in section 3. However, using the classification results of section 3 from the very beginning considerably simplifies the calculations.

Remark 2. We point out that the formulae for dispersive deformations contain the expression

$$\eta_w\mu^3 + (\psi_w + \eta_u)\mu^2 + (\varphi_w + \psi_u)\mu + \varphi_u$$

in the denominator. Since μ is assumed to be arbitrary, this expression is nonzero unless φ, ψ, η satisfy the relations

$$\eta_w = 0, \qquad \psi_w + \eta_u = 0, \qquad \varphi_w + \psi_u = 0, \qquad \varphi_u = 0. \tag{50}$$

These relations characterize the so-called *totally linearly degenerate systems*, which are known to be quite special from the point of view of the global existence of classical solutions: it was conjectured in [23] that smooth initial data for totally linearly degenerate systems do not break down in finite time. Modulo the integrability conditions (36), the relations (50) lead to equations of the form

$$u_t = \alpha(wu_x - uw_x) + \beta(wu_y - uw_y) + \gamma w_y, \qquad w_x = u_y,$$

which have been discussed before in the context of the so-called 'universal hierarchy' [27]. For totally linearly degenerate systems (in particular, for linear systems), the procedure based on deformations of hydrodynamic reductions does not work, as the following simple example shows. Let us consider the KP equation,

$$u_t = \alpha uu_x + w_y + \epsilon^2 u_{xxx}, \qquad w_x = u_y,$$

where we introduced a parameter α: for $\alpha = 0$ the equation becomes linear. Looking for deformed 1-phase solutions in the form

$$u = R, \qquad w = w(R) + \epsilon^2(\cdots) + O(\epsilon^4),$$

where

$$R_y = \mu R_x + \epsilon^2(\cdots) + O(\epsilon^4), \qquad R_t = (\mu^2 + \alpha R)R_x + \epsilon^2(\cdots) + O(\epsilon^4),$$

one can obtain the relation $\alpha b(R) - \mu' = 0$, where $b(R)$ is the coefficient at R_{xxx} in the ϵ^2-term in the expansion of R_y. For $\alpha = 0$, one cannot solve for $b(R)$, and obtains a relation $\mu' = 0$. Thus, the linear equation $u_t = w_y + \epsilon^2 u_{xxx}$ does not inherit generic hydrodynamic reductions of its dispersionless limit. Another example of this kind is provided by the potential KP equation,

$$u_t = w_y + \frac{\epsilon}{2}u_x^2 + \epsilon^2 u_{xxx}. \tag{51}$$

One can show that this equation does not inherit hydrodynamic reductions of its dispersionless limit. However, some particular reductions can be inherited, for instance, those with $\mu = \text{const}$.

Thus, we exclude totally linearly degenerate systems from the further considerations: dispersive deformations of such systems do not inherit hydrodynamic reductions, and require a different approach.

4.1. Non-symmetric dispersive equations

In this section, we summarize the classification results for integrable non-symmetric third-order equations (1),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\cdots) + \epsilon^2(\cdots), \qquad w_x = u_y,$$

which are obtained by adding dispersive terms to integrable dispersionless candidates from section 3.1. Thus, we follow the classification of section 3.1.

Case 1: We have verified that the exponential solutions (39) do not survive, so that all non-trivial examples come from the polynomial case (38),

$$\eta = 1, \qquad \psi = \gamma u, \qquad \varphi = \beta w - \tfrac{1}{2}\beta(\beta+\gamma)u^2 + \delta u.$$

We point out that the corresponding dispersionless system possesses the Lax pair

$$S_y = \beta u S_x + r(S_x), \qquad S_t = \left(\beta w + \tfrac{1}{2}\beta(\beta+\gamma)u^2\right)S_x + \beta u S_x r'(S_x) + z(S_x), \tag{52}$$

where

$$r(S_x) = -\frac{\delta}{\beta+\gamma}S_x + S_x^{\frac{2\beta+\gamma}{\beta}}, \qquad z' = r'^2.$$

Lax pairs of this kind, consisting of two compatible Hamilton–Jacobi type equations, were first introduced by Zakharov in [37]. A detailed analysis of dispersive deformations leads to the two branches: $\gamma = 0$, which corresponds to the $(2+1)$-dimensional Gardner equation (example 3 of section 2.1), and the case $\gamma = -3\beta$. In the latter case, one can set $\delta = 0$, which leads to the apparently new equation (5),

$$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2\left[B^3(u) - \beta B^2(u)u_x\right],$$

where $B = \beta u D_x - D_y$. The dispersionless limit of this equation possesses the Lax pair

$$S_x S_y = \beta u S_x^2 + \tfrac{1}{3}, \qquad S_t = \beta^3 u^3 S_x^3 - S_y^3 + \beta w S_x, \tag{53}$$

which follows from (52) when $\gamma = -3\beta$. Its dispersive extension is

$$\psi_{xy} = \beta u \psi_{xx} + \frac{1}{3\epsilon^2}\psi, \qquad \psi_t = \beta^3\epsilon^2 u^3\psi_{xxx} - \epsilon^2\psi_{yyy} + 3\beta^2\epsilon^2 u u_y \psi_{xx} + \beta w \psi_x. \tag{54}$$

This is case (5) from section 1.

Case 2: One can prove that none of the logarithmic cases (41), (42) and (43) survive, so that all non-trivial examples come from the power case (40),

$$\eta = u, \qquad \psi = \alpha w + \gamma u^{\alpha+1}, \qquad \varphi = \delta u^{2\alpha+1}.$$

Further analysis leads to the following branches.

Subcase 2.1: $\alpha = 1$. In this case

$$\eta = u, \qquad \psi = w + \gamma u^2, \qquad \varphi = \delta u^3.$$

The corresponding dispersionless Lax pair is of the form

$$S_y = ua, \qquad S_t = uwa + \tfrac{1}{3}a(\gamma + a')u^3, \tag{55}$$

where the function $a(S_x)$ solves the ODE $aa'' - 2a'^2 = 3\delta + 2\gamma a'$. The further analysis gives either $\gamma = \delta = 0$, which leads to the non-symmetric VN cases (examples 4 and 5 of section 2.1, in this case one can take $a = 1/S_x$), or $\delta = \frac{4}{27}\gamma^2$, in which case one arrives at the apparently new dispersive equation (6),

$$u_t = \tfrac{4}{27}\gamma^2 u^3 u_x + (w + \gamma u^2)u_y + u w_y + \epsilon^2\left[B^3(u) - \tfrac{1}{3}\gamma u_x B^2(u)\right],$$

where $B = \frac{1}{3}\gamma u D_x + D_y$. This corresponds to the choice $a = 1/S_x - \frac{\gamma}{3}S_x$ in the dispersionless Lax pair (55), which gives

$$S_x S_y = -\frac{\gamma}{3}uS_x^2 - \frac{u}{3}, \qquad S_t = \frac{\gamma^3}{27}u^3S_x^3 + S_y^3 + \frac{\gamma^2}{27}u^3S_x + wS_y. \tag{56}$$

The dispersive extension of this Lax pair is

$$\begin{aligned}\psi_{xy} &= -\frac{\gamma}{3}u\psi_{xx} - \frac{1}{3\epsilon^2}u\psi,\\ \psi_t &= \frac{\epsilon^2\gamma^3}{27}u^3\psi_{xxx} + \epsilon^2\psi_{yyy} - \frac{\epsilon^2\gamma^2}{3}uu_y\psi_{xx} + \frac{\gamma^2}{27}u^3\psi_x + w\psi_y - \frac{\gamma}{3}uu_y\psi.\end{aligned} \tag{57}$$

The transformation $\gamma \to 3\beta, y \to -y, w \to -w$ reduces this case to equation (6) from section 1.

Subcase 2.2: $\alpha = -2$. In this case, one obtains $\gamma = 0$, while δ can be an arbitrary constant. The corresponding dispersive extension takes the form (7),

$$u_t = \frac{\delta}{u^3}u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u}\left(\frac{1}{u}\right)_{xxx},$$

for $\delta = 0$ it reduces to the HD equation (example 9 of section 2.1). The dispersionless limit of this equation possesses the Lax pair

$$S_y = \frac{S_x^2 + \tau}{u^2}, \qquad S_t = -2w\frac{S_x^2+\tau}{u^2} + \frac{4}{3}\frac{S_x^3 + \tau S_x}{u^3}; \tag{58}$$

here $\tau = 3\delta/4$. Its dispersive extension is of the form $L_t = [A, L]$, where

$$\begin{aligned}L &= \frac{\epsilon^2}{u^2}D_x^2 + \frac{\epsilon}{\sqrt{3}}D_y + \frac{\delta}{4u^2},\\ A &= \frac{4\epsilon^2}{u^3}D_x^3 + \left(-\frac{6\epsilon^2 u_x}{u^4} + \frac{2\sqrt{3}\epsilon w}{u^2}\right)D_x^2 + \frac{\delta}{u^3}D_x + \left(-\frac{3\delta u_x}{2u^4} + \frac{\sqrt{3}\delta w}{2\epsilon u^2}\right).\end{aligned} \tag{59}$$

Case 3: One can show that none of the examples from this class possess third-order dispersive extensions.

4.2. Symmetric dispersive equations

A detailed analysis of dispersive extensions of the form (2),

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \tau v_x + \epsilon(\cdots) + \epsilon^2(\cdots), \qquad w_x = u_y, v_y = u_x,$$

does not give any new examples: everything reduces to the two cases of section 2.2. Note that both symmetric VN and mVN equations can be viewed as linear combinations of the two commuting non-symmetric counterparts thereof.

5. Concluding remarks

We have proposed a new approach to the classification of integrable equations in 2 + 1 dimensions based on the concept of hydrodynamic reductions and their dispersive deformations. It consists of the two steps:

- classification of dispersionless systems which may (potentially) arise as dispersionless limits of soliton equations. This can be efficiently achieved using the method of hydrodynamic reductions as outlined in [10];

- classification of possible dispersive deformations based on the requirement that hydrodynamic reductions of the dispersionless limit be inherited by the dispersive equation [14].

This procedure was applied to the classification of third-order soliton equations with 'simplest' non-localities. Further research in this direction may include the following topics:

(a) Classification of more general (in particular, higher order) soliton equations/systems with more complicated structure of non-local terms. Thus, one may allow 'nested' non-localities of type $w = D_x^{-1} D_y u$, $v = D_x^{-1} D_y F(u, w)$, etc.
(b) Construction of dispersive deformations via an appropriate quantization of the corresponding dispersionless Lax pairs [37].
(c) Investigation of the structure of multi-soliton solutions of the new equations (5)–(7) in the spirit of [5, 6].

Acknowledgments

We thank B Dubrovin, A Hone, B Konopelchenko, A Mikhailov, M Pavlov and J P Wang for clarifying discussions. We also thank the referees for useful comments. The research of EVF and AM was supported by the EPSRC grant EP/D036178/1, the European Union through the FP6 Marie Curie RTN project ENIGMA (contract number MRTN-CT-2004-5652), and the ESF programme MISGAM. The research of VSN was supported by the EPSRC Postdoctoral Fellowship grant EP/C527747/1.

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